An alternative J_2 material model with isotropic hardening for coupled thermal-structural finite-strain elastoplastic analyses

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Abstract. In this paper an alternative J_2 material model with isotropic hardening for finite-strain elastoplastic analyses is presented. The model is based on a new non-linear continuum mechanical theory of finite deformations of elastoplastic media which allows us to describe the plastic flow in terms of various instances of the yield surface and corresponding stress measures in the initial and current configurations of the body. The approach also allows us to develop thermodynamically consistent material models in every respect. Consequently, the models not only do comply with the principles of material modelling, but also use constitutive equations, evolution equations and even 'normality rules' during return mapping which can be expressed in terms of power conjugate stress and strain measures or their objective rates. Therefore, such models and the results of the analyses employing them no longer depend on the description and the particularities of the material model formulation. Here we briefly present an improved version of our former material model capable of modelling ductile-to brittle failure mode transition and demonstrate the model in a numerical example using a fully coupled thermal-structural analysis.

Keywords: finite-strain formulation, thermodynamically consistent formulation, thermal-structural finite element analysis, strong coupling, generalized J_2 plasticity, isotropic hardening, material damping, ductile-to-brittle failure mode transition

1 Introduction

Modelling materials within the framework of finite-strain thermoelastoplasticity represents a challenging task in computational mechanics. While plastic behaviour of structural materials within the framework of small-strain thermoelastoplasticity is now well understood, due to the fact, that small-strain flow plasticity theories work well and their results are in agreement with experiments, the same cannot be said for finite-strain flow plasticity theories [1]. Although innumerable material models for finite-strain elastoplasticity have by now been proposed [2-10], such models in general lack universality, as their analysis results depend on the description used in the model and the particularities of the model formulation. The modelling method might simply need some

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developments in the non-linear continuum theory of finite deformations of elastoplastic media in order that the related theories could be considered to be complete.

Contemporary flow plasticity theories in finite-strain phenomenological thermoelastoplasticity are either based on an additive decomposition of a strain rate tensor into an elastic part, a plastic part and a thermal part, or the multiplicative decomposition of a deformation gradient into an elastic part, a plastic part and a thermal part. The first type of theories are considered to be ad hoc extensions of infinitesimal flow plasticity theories into the area of finite deformations of elastic media to cover large displacements, but small strains of the deforming body. The related material models use an additive decomposition of a strain rate tensor into an elastic part, a plastic part and a thermal part and are based on a hypoelastic stress-strain relationship while utilizing the nonlinear continuum mechanical theory of elastic media to describe the plastic behaviour of the material [2, 11-16].

The second type of flow plasticity theories are now generally accepted as "proper theories", utilizing the theory of single-crystal plasticity, to describe the micromechanics of irreversible deformations in the material. The related material models use a multiplicative split of a deformation gradient into an elastic part, a plastic part and the thermal part, the classical flow plasticity models from small-strain elastoplasticity while utilizing the nonlinear continuum mechanical theory of elastic media to describe the plastic behaviour of the material [2, 3, 17-23].

Our ongoing research however has shown, that both types of the aforementioned theories are just variants of our modified nonlinear continuum theory of finite deformations of elastoplastic media, using an additive decomposition of the displacement field into an elastic part, a plastic part and a thermal part, which describes the plastic flow in terms of various instances of a yield surface and stress measures in the initial or current configuration of the body. Moreover, the theory allows for the generalization of the contemporary flow plasticity theories and the development of alternative material models which are thermodynamically consistent in every respect. As a result, the analysis results of the models no longer depend on the particularities of the model formulation. We will show that the contemporary strain rate tensor additive decomposition based theories are in fact finite-strain theories, but they are constrained when the material model in them is defined in terms of a Cauchy stress tensor based yield surface in the current configuration of the body. That is why they appear as if they had mixed finite-strain small-strain formulations. The contemporary deformation gradient multiplicative split based theories on the other hand are rather incomplete and not quite consistent with the theory of nonlinear continuum mechanics, because they neglect the displacement fields in the definition of the deformation gradient and their parts. Moreover, the deformation gradient cannot have a Lagrangian form unless the Lagrangian displacement field has an additive decomposition. These are a few major implications of our nonlinear continuum theory of finite deformations of elastoplastic media.

We will not present the non-linear continuum theory of elastoplastic media herein, nor will we explore any of its part. Our aim in this paper is to present an alternative J_2 material model with isotropic hardening, which is the only model at present that meets the requirement of thermodynamic consistency. Here we briefly present a modified version of our former material model capable of imitating ductile-to brittle failure mode transition and demonstrate the model in a numerical example using a fully coupled thermal-structural analysis.

2 Theory

We will use Lagrangian description to describe the motion of a material particle of a deformable body. Though a single form is sufficient to define a thermodynamically consistent material model, we will provide all forms of the material model in terms of different stress and strain measures or their objective rates respectively, as similar forms cannot be found anywhere in the contemporary scientific literature.

2.1 Kinematics of the deformation

In order to describe the kinematics of deformation of an elastoplastic media, we assume that the material/Lagrangian displacement field can additively be decomposed into an elastic part, a plastic part and a thermal part $\mathbf{u} = \mathbf{u}^{el} + \mathbf{u}^{pl} + \mathbf{u}^{th}$, where

$$\mathbf{u}^{th} = \left[\int_{X_{ref}}^{X} \alpha_X \cdot \left[T(\mathbf{X},t) - T_{ref} \right] \cdot dX, \quad \int_{Y_{ref}}^{Y} \alpha_Y \cdot \left[T(\mathbf{X},t) - T_{ref} \right] \cdot dY, \quad \int_{Z_{ref}}^{Z} \alpha_Z \cdot \left[T(\mathbf{X},t) - T_{ref} \right] \cdot dZ \right]^T.$$

Here $\alpha_X, \alpha_Y, \alpha_Z$ denote the coefficients of thermal expansion defined in the initial configuration of the body and $T(\mathbf{X},t), T_{ref}$ are the material temperature field and the reference temperature respectively. In this case neither the Green strain tensor $\mathbf{E} = 1/2 \cdot \left(\mathbf{F}^T \cdot \mathbf{F} - \mathbf{I} \right)$, nor the Almansi strain tensor $\mathbf{e} = 1/2 \cdot \left(\mathbf{I} - \mathbf{F}^{-T} \cdot \mathbf{F}^{-1} \right)$ has a decomposition into an elastic part, a plastic part and a thermal part, but an additive decomposition exists when one evaluates the objective time derivatives of the tensors. The material $\dot{\mathbf{E}}$ and the spatial $\mathbf{d} = \mathcal{L}_e(\mathbf{e})$ strain-rate tensors then take the following forms

$$\dot{\mathbf{E}} = \frac{1}{2} \cdot \left(\dot{\mathbf{F}}^T \cdot \mathbf{F} + \mathbf{F}^T \cdot \dot{\mathbf{F}} \right) = \dot{\mathbf{E}}^{el} + \dot{\mathbf{E}}^{pl} + \dot{\mathbf{E}}^{th}, \tag{1}$$

where

$$\dot{\mathbf{E}}^{el} = \frac{1}{2} \cdot \left[\left(\frac{\partial \dot{\mathbf{u}}^{el}}{\partial \mathbf{X}} \right)^T \cdot \mathbf{F} + \mathbf{F}^T \cdot \frac{\partial \dot{\mathbf{u}}^{el}}{\partial \mathbf{X}} \right], \tag{2}$$

$$\dot{\mathbf{E}}^{pl} = \frac{\dot{\lambda}}{2} \cdot \left[\left(\frac{\partial^P \Psi}{\partial \mathbf{P}} \right)^T \cdot \mathbf{F} + \mathbf{F}^T \cdot \frac{\partial^P \Psi}{\partial \mathbf{P}} \right], \tag{3}$$

$$\dot{\mathbf{E}}^{th} = \frac{1}{2} \left[\begin{pmatrix} \alpha_X & 0 & 0 \\ 0 & \alpha_Y & 0 \\ 0 & 0 & \alpha_Z \end{pmatrix} \cdot \mathbf{F} + \mathbf{F}^T \cdot \begin{pmatrix} \alpha_X & 0 & 0 \\ 0 & \alpha_Y & 0 \\ 0 & 0 & \alpha_Z \end{pmatrix} \right] \dot{T}, \quad \frac{\partial \dot{\mathbf{u}}^{th}}{\partial \mathbf{X}} = \begin{pmatrix} \alpha_X & 0 & 0 \\ 0 & \alpha_Y & 0 \\ 0 & 0 & \alpha_Z \end{pmatrix} \dot{T}, \tag{4}$$

$$\frac{\partial \dot{\mathbf{u}}^{pl}}{\partial \mathbf{X}} = \dot{\lambda} \cdot \frac{\partial^P \Psi}{\partial \mathbf{P}}, \quad \text{and} \quad \frac{\partial^P \Psi}{\partial \mathbf{P}} \neq \left(\frac{\partial^P \Psi}{\partial \mathbf{P}} \right)^T, \tag{5}$$

$$\mathbf{d} = \mathbf{d}^{el} + \mathbf{d}^{pl} + \mathbf{d}^{th}, \quad \mathbf{d}^{el} = \mathbf{F}^{-T} \cdot \dot{\mathbf{E}}^{el} \cdot \mathbf{F}^{-1}, \quad \mathbf{d}^{pl} = \mathbf{F}^{-T} \cdot \dot{\mathbf{E}}^{pl} \cdot \mathbf{F}^{-1}, \quad \mathbf{d}^{th} = \mathbf{F}^{-T} \cdot \dot{\mathbf{E}}^{th} \cdot \mathbf{F}^{-1}. \tag{6}$$

Here $\mathbf{X}$ denotes the position vector of a material point and $\mathbf{x} = \mathbf{X} + \mathbf{u}$ is the position vector of the corresponding spatial point. The deformation gradient $\mathbf{F} = \mathbf{I} + \partial \mathbf{u} / \partial \mathbf{X} = \mathbf{I} + \partial \mathbf{u}^{el} / \partial \mathbf{X} + \partial \mathbf{u}^{pl} / \partial \mathbf{X} + \partial \mathbf{u}^{th} / \partial \mathbf{X}$ can then be expressed either as a function of the

material displacement field $\mathbf{u}$ alone, or as a function of its elastic $\mathbf{u}^{el}$, plastic $\mathbf{u}^{pl}$ and thermal $\mathbf{u}^{th}$ parts. The symbols $\dot{\mathbf{E}}^{el}, \dot{\mathbf{E}}^{pl}, \dot{\mathbf{E}}^{th} / \mathbf{d}^{el}, \mathbf{d}^{pl}, \mathbf{d}^{th}$ denote the elastic, the plastic and the thermal material/spatial strain rate tensors, in which the plastic flow is defined by Eqn. $(5)_1$ as a product of a plastic multiplier $\dot{\lambda}$ and an appropriate yield surface normal $\partial^P \Psi / \partial \mathbf{P}$ in terms of the 1st Piola-Kirchhoff stress tensor $\mathbf{P}$. Please note that we have also simplified the calculation of the material thermal strain rate tensor (Eqn. $(4)_1$) in which we neglected the off-diagonal elements of the material gradient of the thermal velocity field (Eqn. $(4)_2$) as their contribution is small unless the temperature gradient is high within the element. Here the symbol $\mathcal{L}_e(\bullet)$ denotes the Lie derivative operator $\mathcal{L}_e(\bullet) = \mathbf{F}^{-T} \cdot \left[\partial\left(\mathbf{F}^T \cdot (\bullet) \cdot \mathbf{F}\right) / \partial t\right] \cdot \mathbf{F}^{-1}$ of a spatial strain tensor. Please note, that the elastic, the plastic and the thermal strain rate tensors have similar forms as the strain rate tensor itself. Furthermore, it can be shown, that the plastic flow defined by Eqn. $(5)_1$ is not constrained, resulting in Eqns. (3) and $(6)_3$ respectively, as the only non-degenerated forms of the material and spatial plastic strain rate tensors.

2.2 The constitutive equation of the material

Proper formulation of a material model for finite-strain thermoelastoplasticity allows for the definition of a constitutive equation of the material in terms of various stress and strain measures or their objective rates in both the body's initial and current configurations. As a result, the constitutive equation of a material cannot be unique, but has to have various forms. These however have to comply with the principles of material modelling, particularly meet the requirements of material objectivity and moreover be thermodynamically consistent in order that they defined the same material. Furthermore, because the additive decompositions defined by Eqns. (1), $(6)_1$ exist in rate forms only, the constitutive equation too must have rate forms. In fact, Eqns. (7)-(10) define a true hypoelastic based thermoelastoplastic material model, which does not have a form in terms of finite strain measures.

In this research we have modified our former material model capable of imitating ductile-to-brittle failure mode transition in a ductile material at high strain rates [24]. In agreement with the above, the rate form of the constitutive equation of the material can take any of the following forms:

$$\dot{\mathbf{S}} = {}^{mat}\mathbf{C}^{el} : \left(\dot{\mathbf{E}} - \dot{\mathbf{E}}^{th} - x \cdot \dot{\mathbf{E}}^{pl}\right) + {}^{mat}\mathbf{C}^{vis} : \left[\ddot{\mathbf{E}} - (1-x) \cdot \ddot{\mathbf{E}}^{pl}\right], \tag{7}$$

$$\mathcal{L}_P(\mathbf{P}) = \mathbf{F} \cdot \dot{\mathbf{S}} = \mathbf{F} \cdot \left\{ {}^{mat}\mathbf{C}^{el} : \left(\dot{\mathbf{E}} - \dot{\mathbf{E}}^{th} - x \cdot \dot{\mathbf{E}}^{pl}\right) + {}^{mat}\mathbf{C}^{vis} : \left[\ddot{\mathbf{E}} - (1-x) \cdot \ddot{\mathbf{E}}^{pl}\right] \right\}, \tag{8}$$

$$\mathcal{L}_O(\boldsymbol{\tau}) = \mathbf{F} \cdot \dot{\mathbf{S}} \cdot \mathbf{F}^T = J \cdot {}^{spat}\mathbf{C}^{el} : \left(\mathbf{d} - \mathbf{d}^{th} - x \cdot \mathbf{d}^{pl}\right) + J \cdot {}^{spat}\mathbf{C}^{vis} : \left[\mathcal{L}_e(\mathbf{d}) - (1-x) \cdot \mathcal{L}_e\left(\mathbf{d}^{pl}\right)\right], \tag{9}$$

$$\mathcal{L}_T(\boldsymbol{\sigma}) = J^{-1} \cdot \mathbf{F} \cdot \dot{\mathbf{S}} \cdot \mathbf{F}^T = {}^{spat}\mathbf{C}^{el} : \left(\mathbf{d} - \mathbf{d}^{th} - x \cdot \mathbf{d}^{pl}\right) + {}^{spat}\mathbf{C}^{vis} : \left[\mathcal{L}_e(\mathbf{d}) - (1-x) \cdot \mathcal{L}_e\left(\mathbf{d}^{pl}\right)\right], \tag{10}$$

where $${}^{mat}\mathbf{C}^{el} = 2 \cdot G \cdot \mathbf{I} + \lambda^{el} \cdot \mathbf{1} \otimes \mathbf{1}, \quad {}^{mat}\mathbf{C}^{vis} = 2 \cdot G^{vis} \cdot \mathbf{I} + \lambda^{vis} \cdot \mathbf{1} \otimes \mathbf{1}, \quad G = \frac{E}{2 \cdot (1+\nu)}, \tag{11}$$

$$\lambda^{el} = \frac{\nu \cdot E}{(1+\nu) \cdot (1-2 \cdot \nu)}, \; G^{vis} = \frac{E^{vis}}{2 \cdot \left(1+\nu^{vis}\right)}, \; \lambda^{vis} = \frac{\nu^{vis} \cdot E^{vis}}{\left(1+\nu^{vis}\right) \cdot \left(1-2 \cdot \nu^{vis}\right)}, \tag{12}$$

$${}^{spat}\mathbf{C}^{el}_{ijkl} = J^{-1} \cdot F_{im} \cdot F_{jn} \cdot F_{ko} \cdot F_{lp} \cdot {}^{mat}\mathbf{C}^{el}_{mnop}, \tag{13}$$

$${}^{spat}\mathbf{C}^{vis}_{ijkl} = J^{-1} \cdot F_{im} \cdot F_{jn} \cdot F_{ko} \cdot F_{lp} \cdot {}^{mat}\mathbf{C}^{vis}_{mnop}. \tag{14}$$

In Eqns. (7)-(14) the symbols $\mathbf{S},\mathbf{P},\boldsymbol{\tau},\boldsymbol{\sigma},\dot{\mathbf{S}},\mathcal{L}_P\left(\mathbf{P}\right),\mathcal{L}_O\left(\boldsymbol{\tau}\right),\mathcal{L}_T\left(\boldsymbol{\sigma}\right)$ denote the 2nd Piola-Kirchhoff stress tensor, the 1st Piola-Kirchhoff stress tensor, the Kirchhoff stress tensor, the Cauchy stress tensor and their objective rates respectively. They are the time derivative of the 2nd Piola-Kirchhoff stress tensor $\dot{\mathbf{S}}$, the Lie derivative of the 1st Piola-Kirchhoff stress tensor $\mathcal{L}_P\left(\mathbf{P}\right)$, defined in terms of the Lie derivative operator of a mixed spatial-material stress tensor $\mathcal{L}_P\left(\bullet\right)=\mathbf{F}\cdot\left[\partial\left(\mathbf{F}^{-1}\cdot\left(\bullet\right)\right)/\partial t\right]$, the Oldroyd rate of the Kirchhoff stress $\mathcal{L}_O\left(\boldsymbol{\tau}\right)$ defined in terms of the Lie derivative operator of a spatial stress tensor $\mathcal{L}_O\left(\bullet\right)=\mathbf{F}\cdot\left[\partial\left(\mathbf{F}^{-1}\cdot\left(\bullet\right)\cdot\mathbf{F}^{-T}\right)/\partial t\right]\cdot\mathbf{F}^{T}$ and the Truesdell rate of the Cauchy stress $\mathcal{L}_T\left(\boldsymbol{\sigma}\right)$ defined in terms of the Truesdell derivative operator of a spatial stress tensor $\mathcal{L}_T\left(\bullet\right)=J^{-1}\cdot\mathbf{F}\cdot\left[\partial\left(J\cdot\mathbf{F}^{-1}\cdot\left(\bullet\right)\cdot\mathbf{F}^{-T}\right)/\partial t\right]\cdot\mathbf{F}^{T}$. The latter operator actually carries out Lie differentiation, but with rearranged terms of its final form. Here the fourth order material elasticity tensor ${}^{mat}\mathbf{C}^{el}$ and the fourth order material viscosity tensor ${}^{mat}\mathbf{C}^{vis}$ have similar forms as the fourth order elasticity tensor of the St.-Venant Kirchhoff material [25], using two independent material parameters E,ν and E^{vis},ν^{vis} respectively. The fourth order spatial elasticity and viscosity tensors ${}^{spat}\mathbf{C}^{el}$, ${}^{spat}\mathbf{C}^{vis}$ the can be determined in accordance with Eqns. (13) and (14), where $J=\det\left(\mathbf{F}\right)$ is the Jacobian of the deformation. The variable x denotes the ratio of ductile and total damage increment [24]. Please also note that the objective rates $\dot{\mathbf{S}},\mathcal{L}_P\left(\mathbf{P}\right),\mathcal{L}_O\left(\boldsymbol{\tau}\right),\mathcal{L}_T\left(\boldsymbol{\sigma}\right)$ transform in the same way from one form to another as do the stress tensors $\mathbf{S},\mathbf{P},\boldsymbol{\tau},\boldsymbol{\sigma}$, which ensure that the formulation is thermodynamically consistent.

2.3 On the thermodynamic consistency of the formulation

It is essential that the formulation of the material model be thermodynamically consistent, as it ensures that the analysis results of the model is independent of the description and the particularities of the model formulation. The thermodynamic consistency of the formulation is then guaranteed by the appropriate transformations mentioned in the above and by the following equations ensuring the equivalence of the rate of change of the internal elastic deformation energy Eqn. (15), the internal thermal deformation energy Eqn. (16) and the internal plastic deformation energy Eqn. (17)

$$\dot{\mathbf{E}}^{el}:\mathbf{S}\cdot dV_0=\frac{\partial\dot{\mathbf{u}}^{el}}{\partial\mathbf{X}}:\mathbf{P}\cdot dV_0=\mathbf{d}^{el}:\boldsymbol{\tau}\cdot dV_0=\mathbf{d}^{el}:\boldsymbol{\sigma}\cdot dv, \tag{15}$$

$$\dot{\mathbf{E}}^{th}:\mathbf{S}\cdot dV_0=\frac{\partial\dot{\mathbf{u}}^{th}}{\partial\mathbf{X}}:\mathbf{P}\cdot dV_0=\mathbf{d}^{th}:\boldsymbol{\tau}\cdot dV_0=\mathbf{d}^{th}:\boldsymbol{\sigma}\cdot dv, \tag{16}$$

$$\dot{\mathbf{E}}^{pl}:\mathbf{S}\cdot dV_0=\frac{\partial\dot{\mathbf{u}}^{pl}}{\partial\mathbf{X}}:\mathbf{P}\cdot dV_0=\mathbf{d}^{pl}:\boldsymbol{\tau}\cdot dV_0=\mathbf{d}^{pl}:\boldsymbol{\sigma}\cdot dv, \tag{17}$$

which eventually ensure the equivalence of the rate of change of the overall internal deformation energy [26].

2.4 Mathematical modelling of the plastic flow

Similarly as in the case of the rate forms of the constitutive equation of the material, proper formulation of a finite-strain flow plasticity theory allows for the description of the plastic flow in terms of various instances of a yield surface and corresponding stress measures in either the initial or current configuration of the body. Let the instances of the yield surface be defined as ${}^{S}\Psi = {}^{S}\Psi\left(\mathbf{S},\mathbf{q}\right)$, ${}^{P}\Psi = {}^{P}\Psi\left(\mathbf{P},\mathbf{q}\right)$, ${}^{\tau}\Psi = {}^{\tau}\Psi\left(\boldsymbol{\tau},\mathbf{q}\right)$, ${}^{\sigma}\Psi = {}^{\sigma}\Psi\left(\boldsymbol{\sigma},\mathbf{q}\right)$ in terms of the stress measures $\mathbf{S},\mathbf{P},\boldsymbol{\tau},\boldsymbol{\sigma}$ and a vector of internal variables $\mathbf{q}$. After introducing similar kinematic equations as the ones defined by Eqns. (1)-(6), they serve a basis for the first nonlinear continuum mechanical theory of finite deformations of elastoplastic media. Moreover, since they define the same admissible stress space and the same plastic flow respectively, the instances of the yield surface cannot be independent of each other. In fact ${}^{S}\Psi, {}^{P}\Psi, {}^{\tau}\Psi, {}^{\sigma}\Psi$ are related by the following formulas

$$\frac{\partial {}^{P}\Psi}{\partial \mathbf{P}}\cdot\mathbf{F}^{-1} = \frac{\partial {}^{\sigma}\Psi}{\partial \boldsymbol{\sigma}}, \quad \frac{\partial {}^{P}\Psi}{\partial \mathbf{P}}\cdot\mathbf{F}^{-1} = \frac{\partial {}^{\tau}\Psi}{\partial \boldsymbol{\tau}}, \quad \mathbf{F}^{T}\cdot\frac{\partial {}^{P}\Psi}{\partial \mathbf{P}} = \frac{\partial {}^{S}\Psi}{\partial \mathbf{S}}. \tag{18}$$

Furthermore, one of the yield surfaces ${}^{S}\Psi, {}^{P}\Psi, {}^{\tau}\Psi, {}^{\sigma}\Psi$ has to be chosen as a reference yield surface to define the material model. It can be shown, that when ${}^{\sigma}\Psi$ or ${}^{\tau}\Psi$ is chosen as a reference yield surface in the current configuration of the body, we recover the contemporary flow plasticity theories. Moreover, please note that Eqns. $(18)_1$ and $(18)_2$, represent constraint equations, making the contemporary additive decomposition based theories appear as if they had mixed finite-strain-small-strain formulations.

Crucial part in finite-strain material modelling is thermodynamic consistency of the plastic flow. It ensures, that plastic deformations in the material are independent of the description and the particularities of the model formulation. The thermodynamic consistency of the plastic flow can then be expressed as follows

$$\dot{\mathbf{E}}^{pl} : \dot{\mathbf{S}}\cdot dV_0 = \frac{\partial \dot{\mathbf{u}}^{pl}}{\partial \mathbf{X}} : \mathcal{L}_P\left(\mathbf{P}\right)\cdot dV_0 = \mathbf{d}^{pl} : \mathcal{L}_O\left(\boldsymbol{\tau}\right)\cdot dV_0 = \mathbf{d}^{pl} : \mathcal{L}_T\left(\boldsymbol{\sigma}\right)\cdot dv, \tag{19}$$

where dV_0 is an infinitesimal volume element in the body's initial configuration and $dv = J\cdot dV_0$ is its spatial counterpart. In terms of the above one can prove, that Eqn. (19) has an equivalent form which can be expressed as follows

$$\frac{\partial {}^{S}\Psi}{\partial \mathbf{S}} : \dot{\mathbf{S}} = \frac{\partial {}^{P}\Psi}{\partial \mathbf{P}} : \mathcal{L}_P\left(\mathbf{P}\right) = \frac{\partial {}^{\tau}\Psi}{\partial \boldsymbol{\tau}} : \mathcal{L}_O\left(\boldsymbol{\tau}\right) = J\cdot\frac{\partial {}^{\sigma}\Psi}{\partial \boldsymbol{\sigma}} : \mathcal{L}_T\left(\boldsymbol{\sigma}\right), \tag{20}$$

which is known as the 'normality rule' and which defines a rate form of a thermodynamically consistent return mapping procedure. The result is of fundamental importance in computational mechanics as it states, how the plastic multiplier ought to be calculated during return mapping when finite-strain elastoplastic or thermoelastoplastic analysis is carried out.

2.5 The reference yield surface

It has been shown in the above, the choice of the reference yield surface governs the material model. As a result, alternative material models can be developed. In our research we have generalized the J_2 flow plasticity theory with isotropic hardening, where we used the ${}^{P}\Psi = {}^{P}\Psi\left(\mathbf{P},\mathbf{q}\right)$, Eqn. $(21)_1$ yield surface, as the reference yield surface, to define our

material model. Please note that the ${}^{P}J_2\left(\mathbf{P}\right)=\mathbf{P}:\mathbf{P}$ invariant no longer bases on the deviatoric part of the 1st Piola-Kirchhoff stress tensor $\mathbf{P}$. This is due to the fact, that the 1st Piola-Kirchhoff stress tensor transforms under the change of the observer as $\mathbf{P}^{+}=\mathbf{Q}_R\cdot\mathbf{P}$, and ${}^{P}J_2\left(\mathbf{P}\right)$ is the only invariant, which is not affected by the change, i.e. ${}^{P}J_2\left(\mathbf{P}\right)={}^{P}J_2\left(\mathbf{P}^{+}\right)$, where $\mathbf{Q}_R$ is an arbitrary rotating tensor expressing the relative rotation of the coordinate systems of the observer with respect to the reference coordinate system. The resulting yield surface is then no longer a cylinder, but a sphere.

$$ {}^{P}\Psi={}^{P}\sigma_{eq}-{}^{P}\sigma_y\le 0,\quad \text{where } {}^{P}\sigma_{eq}={}^{P}\sigma_{eq}\left(\mathbf{P}\right)=\sqrt{{}^{P}J_2\left(\mathbf{P}\right)}=\sqrt{\mathbf{P}:\mathbf{P}}, \tag{21} $$

$$ {}^{P}\sigma_y=F_{UT11}\cdot\sqrt{r^2-\left[a\cdot e^{pl}-\text{center}\right]^2},\ r=\sigma_y+Q,\ \text{center}=\sqrt{r^2-\sigma_y^2}\ \text{ and }\ a=\frac{\text{center}+r}{b}, \tag{22} $$

$$ \dot{e}^{pl}=\dot{e}^{pl}\left(\dot{\mathbf{F}}^{pl}\right)=\sqrt{\dot{\mathbf{F}}^{pl}:\dot{\mathbf{F}}^{pl}}=\dot{\lambda},\ e^{pl}=\int_0^t\dot{e}^{pl}\cdot dt,\ \mathbf{F}^{pl}=\mathbf{I}+\frac{\partial\mathbf{u}^{pl}}{\partial\mathbf{X}},\ \dot{\mathbf{F}}^{pl}=\frac{\partial\dot{\mathbf{u}}^{pl}}{\partial\mathbf{X}}=\dot{\lambda}\cdot\frac{\partial\,{}^{P}\Psi}{\partial\mathbf{P}}. \tag{23} $$

The actual yield stress ${}^{P}\sigma_y$, which is a 1st Piola-Kirchhoff stress measure, determines the radius of the yield surface and is defined by Eqn.$(22)_1$. It is the only nonzero component of a stress tensor $\mathbf{P}_{UT}$ (i.e. ${}^{P}\sigma_y=\left[\mathbf{P}_{UT}\right]_{11}$) coming from an uniaxial tensile test of the modelled material, where the operator $\left[\left(\bullet\right)\right]_{11}$ extracts the element in the first row and the first column of a 2nd order tensor $\left(\bullet\right)$, written as a 3x3 square matrix. The corresponding deformation gradient and the Jacobian of deformation are denoted as $\mathbf{F}_{UT}, J_{UT}$, where $F_{UT11}=\left[\mathbf{F}_{UT}\right]_{11}$ and $J_{UT}=\det\left(\mathbf{F}_{UT}\right)$. Please also note, that the only nonzero element of the corresponding 2nd Piola-Kirchhoff stress tensor $\mathbf{S}_{UT}$, coming from the tensile test of the material is $\left[\mathbf{S}_{UT}\right]_{11}={}^{S}\sigma_y=\sqrt{r^2-\left[a\cdot e^{pl}-\text{center}\right]^2}$. The equation defines an arc of a circle using three material parameters, the constant yield stress of the material σ_y, the maximum stress Q by which the material can harden and the maximum accumulated strain value $b=e^{pl}_{\max}$, at which the material loses its integrity, i.e. ${}^{S}\sigma_y=0$. The relationship between the corresponding stress measures then can be written in tensor form as $\mathbf{P}_{UT}=\mathbf{F}_{UT}\cdot\mathbf{S}_{UT}$, in which the parameters σ_y, Q are 2nd Piola-Kirchhoff stress measures and $e^{pl}\subset\langle 0,b\rangle$. One may note here, that we have also changed the definition of the accumulated plastic strain rate $\dot{e}^{pl}$ (Eqn. $(23)_1$), in the definition of which $\mathbf{F}^{pl}$ denotes the deformation gradient of pure plastic deformations at a particle of the material, whose time derivative is assumed to be in the form of Eqn.$(5)_1$. Other changes in the definitions of the accumulated plastic strain rate $\dot{e}^{pl}$ and the equivalent stress ${}^{P}\sigma_{eq}$ have been needed in order to meet the requirements of thermodynamic consistency in booth a one-dimensional (1D) stress state and a three-dimensional (3D) stress state respectively.

2.6 Calculation of the plastic multiplier

The calculation of the plastic multiplier is a crucial step in finite-strain elastoplastic stress analyses as it determines the value of the stress rate tensor Eqn. (7)-(10), and the plastic part of the strain rate tensors $\dot{\mathbf{E}}^{pl}, \mathbf{d}^{pl}$ during return mapping. Moreover, the return mapping procedure has to be thermodynamically consistent, i.e. it has to comply with Eqn. (20). The condition has not yet been met in any formulation in finite-strain computational plasticity. The thermodynamically consistent return mapping procedure then utilizes the objective differentiation of the yield surface ${}^{P}\Psi$ and it can be expressed as follows

$$\frac{\partial {}^{P}\Psi}{\partial \mathbf{P}} : \mathcal{L}_{\mathrm{P}}\left(\mathbf{P}\right) - \left[\mathcal{L}_{\mathrm{P}}\left(\mathbf{P}_{UT}\right)\right]_{11} = 0, \tag{24}$$

where $\mathcal{L}_{\mathrm{P}}\left(\mathbf{P}\right)$ is then replaced by the rate form of the constitutive equation of the material Eqn. (8), and the second term of Eqn. (24) by the form $\left[\mathcal{L}_{\mathrm{P}}\left(\mathbf{P}_{UT}\right)\right]_{11} = = F_{UT11} \cdot \left\{\left[-a \cdot \left(a \cdot e^{pl} - \mathrm{center}\right)\right] / \sqrt{r^2 - \left[a \cdot e^{pl} - \mathrm{center}\right]^2}\right\} \cdot \dot{e}^{pl}$. Please also note, that the first term of Eqn. (24) can be replaced by any other term of Eqn. (20), because of the thermodynamic consistency of the formulation.

2.7 The ratio of ductile and total damage increment

The idea of the ratio of ductile and total damage increment x was first introduced by Écsi and Élesztős in order to take into account the internal damping of the material properly during plastic deformations. The ratio allows for the proportional redistribution of the plastic flow between the spring and the damper of a 1D frictional device representing the rheological model of the material [24]. The ratio is determined in an elastic predictor phase during return mapping and its value is then kept constant. Since the return mapping procedure in our material model is carried out in the 1st Piola-Kirchhoff stress space, we had to modify the definition of the ratio as follows

$$x = \frac{\left\langle \mathbf{N} : \mathbf{F} \cdot \left[{}^{mat}\mathbf{C}^{el} : \left(\dot{\mathbf{E}} - \dot{\mathbf{E}}^{th}\right)\right]\right\rangle}{\left\langle \mathbf{N} : \mathbf{F} \cdot \left[{}^{mat}\mathbf{C}^{el} : \left(\dot{\mathbf{E}} - \dot{\mathbf{E}}^{th}\right)\right]\right\rangle + \left\langle \mathbf{N} : \mathbf{F} \cdot \left({}^{mat}\mathbf{C}^{vis} : \ddot{\mathbf{E}}\right)\right\rangle}, \qquad x \in \langle 0,1 \rangle, \tag{25}$$

where

$$\frac{\partial {}^{P}\Psi}{\partial \mathbf{P}} = \mathbf{N}, \quad \mathbf{N} = \frac{\mathbf{P}}{\sqrt{\mathbf{P} : \mathbf{P}}} = \frac{\mathbf{P}}{\|\mathbf{P}\|}, \tag{26}$$

and

$$\langle y \rangle = \frac{y + |y|}{2} \geq 0, \tag{27}$$

denotes the McCauly's brackets, which return zero if $y < 0$ and where $\mathcal{L}_{P}\left(\mathbf{P}\right) = \mathbf{F} \cdot \dot{\mathbf{S}}$. Please also note that all terms of the right-hand-side of Eqn. $(25)_1$ are objective stress rates, so that the value of x is not affected by the change of the observer.

2.8 The heat equation

In order to describe the conservation of heat energy at a particle of the body, we have modified our former heat equation [27]. The heat equation can imitate elastic heating and dissipation induced heating and its material form takes the following form

$$\rho_0 \cdot c_p \cdot \dot{T} + \dot{\mathbf{S}} : \mathbf{E}^{thp} = -\nabla_0 \bullet \mathbf{Q} + R, \qquad R = 0.8 \cdot \left(\mathbf{S}^{el} : x \cdot \dot{\mathbf{E}}^{pl} + \mathbf{S}^{vis} : \dot{\mathbf{E}} \right), \tag{28}$$

where $$\mathbf{E}^{thp} = diag\left[\alpha_X \quad \alpha_Y \quad \alpha_Z\right] \cdot T + \frac{1}{2} diag\left[\alpha_X^2 \quad \alpha_Y^2 \quad \alpha_Z^2\right] \cdot T^2. \tag{29}$$

Here the symbols $\rho_0, c_p, T, \dot{\mathbf{S}}, \mathbf{E}^{thp}, \mathbf{Q}, R$ denote the material density, the specific heat at constant pressure, the absolute temperature, the rate of change of the 2nd Piola-Kirchhoff stress tensor, (Eqn. (7)), a specific Green thermal strain tensor originating from thermal expansion and expressing the thermal strain with respect to the absolute zero temperature, the material heat flux vector and the heat generation rate per unit volume respectively. The elastic heating in the equation is defined by the term $\dot{\mathbf{S}} : \mathbf{E}^{thp}$ and the dissipation induced heating by the heat generation rate per unit volume, where we assumed that 80% of the dissipated mechanical energy changes into heat Eqn. $(28)_2$. Here $\mathbf{S}^{el}$ is the elastic part of the stress tensor and $\mathbf{S}^{vis}$ is its viscous part.

3 Numerical experiment

In our numerical experiment a cross-shaped specimen in biaxial tension was studied. The specimen geometry has been proposed by Müller [28] and it was fabricated from an AlMgSi05 alloy. Table 1 outlines the material properties of the AlMgSi05 alloy specimen used in the finite element analysis.

Table 1. Material properties of the AlMgSi05 alloy specimen

E [Pa]	$6.89 \cdot 10^{10}$
E^{vis} [Pa · s]	$6.89 \cdot 10^{7}$
$\nu = \nu^{vis}$ [-]	0.33
σ_y [Pa]	$100.0 \cdot 10^{6}$
Q [Pa]	$50.0 \cdot 10^{6}$
b [-]	1.0
ρ_0 [kg · m^{-3}]	2700.0
c [J · kg^{-1} · °K^{-1}]	895.0
$k_{XX} = k_{YY} = k_{ZZ}$ [W · m^{-1} · °K^{-1}]	218.0
$\alpha_X = \alpha_Y = \alpha_Z = \alpha$ [°K^{-1}]	$23.4 \cdot 10^{-6}$
h [W · m^{-2} · °K^{-1}]	10.0
ψ [-]	1.0
σ_{EMS} [W · m^{-2} °K^{-4}]	$11.341 \cdot 10^{-9}$

In order to assess the value of the axial component of the deformation gradient coming from the tensile stress of the material F_{UT11}, we solved the one-dimensional (1D) rate form of

the constitutive equation of the material (Eqn. (7)) for the unknown component of the derivative of the axial elastic displacement field with respect to the axial material coordinate $\partial u_x^{el} / \partial X$. The rate form of the constitutive equation of this specific 1D stress analysis, after neglecting the internal damping and the thermal strain in the material, can be expressed in the following finite-strain form

$$ {}^{S}\dot{\sigma}_y = \dot{S}_{11} = E \cdot \left[\frac{\partial \dot{u}_x^{el}}{\partial X} \cdot \left(1 + \frac{\partial u_x^{el}}{\partial X} + \frac{\partial u_x^{pl}}{\partial X} \right) \right], \tag{30} $$

where ${}^{S}\dot{\sigma}_y = \dot{S}_{11}$ is the axial component of the 2nd Piola-Kirchoff stress rate tensor coming from the tensile test of the material and E is the Young's modulus of the material. Furthermore considering that the accumulated plastic strain rate Eqn. $(23)_1$ in this 1D stress state is $\dot{e}^{pl} = \partial \dot{u}_x^{pl} / \partial X = \dot{\lambda}$, and that its integral is $e^{pl} = \partial u_x^{pl} / \partial X$ (Eqn. $(23)_2$), one can find F_{UT11} as a function of the accumulated plastic strain e^{pl} only in the form

$$ F_{UT11} = 1 + \frac{\partial u_x^{el}}{\partial X} + \frac{\partial u_x^{pl}}{\partial X} = 1 + \left[-\left(1 + e^{pl}\right) + \sqrt{\left(1 + e^{pl}\right)^2 + 2 \cdot \frac{{}^{S}\sigma_y}{E}} \right] + e^{pl}, \tag{31} $$

where ${}^{S}\sigma_y = {}^{S}\sigma_y\left(e^{pl}\right)$ see also Eqn. (22).

In the numerical study 1/8 of the specimen body was modelled employing three planes of symmetry. The specimen was loaded at its four ends using $v = 0.84667\ mm/s$ prescribed velocity. Convective and radiation heat transfer was considered through all surfaces, applying $273.15\ ^{\circ}\mathrm{K}$ environmental temperature and radiation source temperature respectively. The bodies were initially at rest with $273.15\ ^{\circ}\mathrm{K}$ initial temperature. The analysis was run as transient-dynamic analysis applying $0.005\ \mathrm{s}$ time step size.

4 Numerical results

Figure 1 shows a few selected results at the end of the finite element analysis, at time 6 s. These are the absolute temperature distribution, the 1st principal stress distribution in terms of the Cauchy's stress measure, the accumulated plastic strain distribution over the current volume of the body and the temperature change time-history at the centre of the cross-shaped specimen respectively.

As can be seen in the figure, the results are very realistic and moreover the temperature change time-history is in agreement with similar temperature change time-histories coming from biaxial tension tests of a cross shaped specimen of an identical material [29]. Moreover, the presented theory is of great importance even from the material testing point of view of ductile materials, as it shows that the contemporary tensile tests are simply not sufficient for finite-strain material property determination without the inclusion of the deformation gradient characterizing the deformation of the tested material during its uniaxial tensile testing. The presented theory thus also serve a basis for improved material testing and it has widespread applications in industry using various manufacturing processes during which finite deformations take place in the processed material [30].

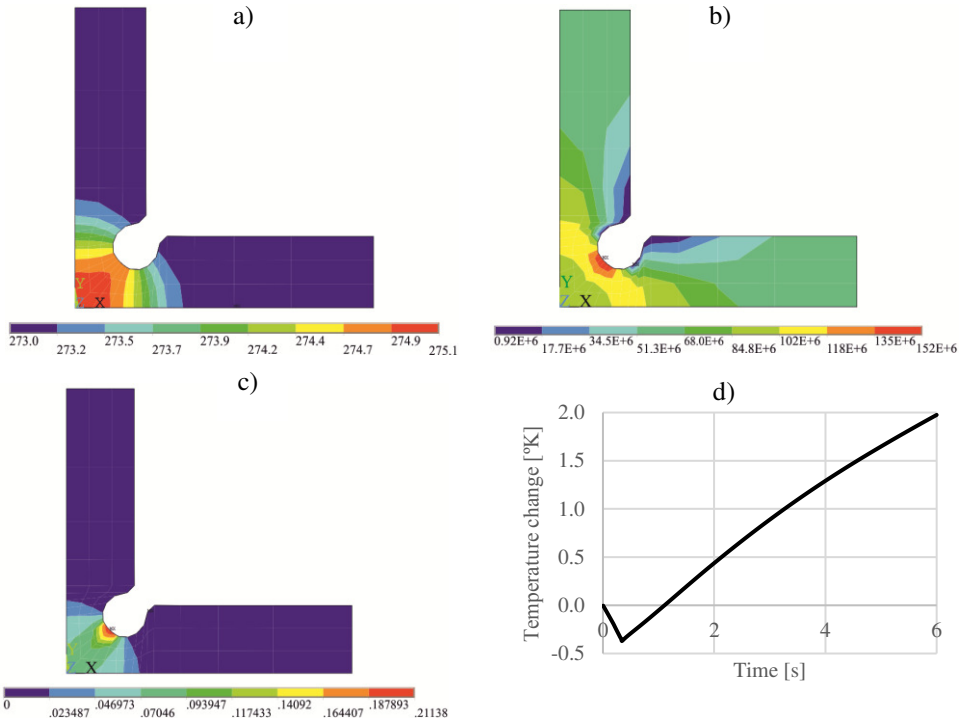

Fig. 1. Selected results from the analysis: **a.** Absolute temperature distribution [°K], **b.** The 1st principal stress distribution in terms of the Cauchy's stress measure [Pa], **c.** Accumulated plastic strain distribution [-], **d.** Temperature change time-history at the centre of the specimen

Conclusions

In this paper an alternative J_2 material model with isotropic hardening for finite-strain elastoplastic analyses was presented. The model is based on a new non-linear continuum mechanical theory of finite deformations of elastoplastic media which allows for the description of the plastic flow in terms of various instances of the yield surface and corresponding stress measures in the body's initial and current configurations. The resulting formulation is thermodynamically consistent, thus the results of the analyses, employing the model, no longer depend on the description and the particularities of the material model formulation. In this study the model behaviour was demonstrated in a numerical example using biaxial tension of a cross-shaped specimen and a fully coupled thermal-structural analysis, the results of which are in agreement with available experiments. The presented theory not only significantly improves upon contemporary thermoelastoplastic analyses, but it also serves as a basis for improved material testing.

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